# Modification of Robson's Algorithm for Finding Maximum Independent Set in Undirected Graph

Igor V. Olemskoy
Saint Petersburg State University,
7/9 Universitetskaya nab., Saint-Petersburg, 199034, Russia
Email: i.olemskoy@spbu.ru

Oksana S. Firyulina
Saint Petersburg State University,
7/9 Universitetskaya nab., Saint-Petersburg, 199034, Russia
Email: o.firyulina@spbu.ru

***Abstract*—The problem of finding the maximum independent set of vertices in an undirected graph is considered. The modification of Robson's algorithm for determining the elements of maximum independent set is proposed.**

## I. Introduction

The problem of finding the maximum independent set of vertices in an undirected graph is one of the most important problems of extremal graph theory which has broad applied value [1], [2]. This problem is one of so-called NP-complete problems - such problems for which there is currently no exact solution algorithms with polynomial estimate of the complexity. Since the number of maximal independent sets is finite, obvious, problem can be solved by constructing all maximal independent set [3]–[5] and then selecting the largest of them. But it's known that the number of maximal independent sets increases exponentially with an increase in the dimensions of the graph [6], consequently enumeration of all the sets for the construction of the largest independent set is inappropriate.

Currently one of the best algorithms for the solution of the problem is the Robson's algorithm [7]. This algorithm is one of the few among a large number of existing exact algorithms ( [8]–[12]) which has a theoretical estimate of complexity. According to this estimation: $O(2^{0.276n})$, where $n$ – number of vertices in a graph, algorithm of Robson is one of the most effective solution to the problem of finding the maximum independent set. This algorithm is focused primarily on the computation of the size of the maximum independent set but in applications it often need to know the elements themselves of this set. In his work Robson pointed out the possibility of modifying the algorithm for finding the elements of the required set, accompanied pseudo-code by appropriate comments. Studied in detail the principles of the algorithm, as well as based on the Robson's comments to the pseudo-code, the authors have carried out the modification of the algorithm, which allows to obtain the required maximum independent set.

## II. Basic definitions

Let $G = (V, E)$ is undirected graph without loops and multiple edges. *The independent set* of vertices in a graph $G = (V, E)$ is a set $J \subseteq V$ :

$$\forall\, p, q \in J \,|\, (p, q) \notin E.$$

The independent set $J \subseteq V$ is *maximal*, if

$$\forall\, v \in V \setminus J \;\; \exists\; \omega \in J : \;(\omega, v) \in E.$$

The maximal independent set of maximum size will be called the *maximum independent set.* The set of vertices $N(v)$ adjacent to a vertex $v \in V$ called a neighborhood of this vertex. The size of set $N(v)$ is degree of vertex $v$ and is denoted $d(v)$. Also, we will consider $N^2(v)$ – the set of vertices adjacent to vertices in the set $N(v)$, except for the vertex $v$:

$$N^2(v) = \cup_{q \in N(v)} N(q) \setminus v,$$

and $\overline{N}(v)$ – the neighborhood of vertex $v$, including vertex $v$:

$$\overline{N}(v) = N(v) \cup \{v\}.$$

## III. Pseudo-code of the algorithm

The pseudo-code of the modified Robson's algorithm is presented below. The numbers mark parts of the algorithm in which the result of a return function is changed.

The main function $ms(G)$:
**function** $ms(G : graph)$ : vertexset;
**begin**
**if** not connected $(G)$ **then**
  **begin**
  $C :=$ smallest connected component of $G$;
  (1) **return** $ms(G{-}C) \cup$ (**if** $|C| \leq 2$ **then** $\{v\}$, $v \in C$ **else** $ms(C)$); { if the smallest connected component $C$ consists of two or less vertices in the independent set included one any vertex $v \in C$}  **end;**
(2) **if** $|G| \leq 1$ **then return** $\{G\}$;
**choose** $A, B$ vertices of $G$ such that
(i) $d(A)$ is minimal and
(ii) $(A, B)$ is an edge of $G$ and $d(B)$ is maximal over all neighbors of vertices with degree $d(A)$;
(3) **if** $d(A) = 1$ **then return** $\{A\} \cup ms(G - \overline{N}(A))$; {The size of the independent set increments by adding in it of the vertex A. Further search of maximal independent set is performed in the graph $G - \overline{N}(A)$}
**if** $d(A) = 2$ **then**
  **begin**
  $B' := N(A) - B$; {the other neighbor of $A$}
  (4) **if** $edge(B, B')$ **then return** $\{A\} \cup ms(G - \overline{N}(A))$;

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(5) $\max_1 := 2 + |ms(G - \overline{N}(B) - \overline{N}(B'))|$;
$\max_2 := 1 + |ms^2(G - \overline{N}(A), N^2(A))|$;
**if** $\max(\max_1, \max_2) = \max_1$
**then return** $\{B, B'\} \cup ms(G - \overline{N}(B) - \overline{N}(B'))$;
**else return** $\{A\} \cup ms^2(G - \overline{N}(A), N^2(A))$; **end;**
(6) **if** $d(A) = 3$ **then** $\max_1 := |ms^2(G - A, N(A))|$;
$\max_2 := 1 + |ms(G - \overline{N}(A))|$;
**if** $\max(\max_1, \max_2) = \max_1$ **then**
**return** $ms^2(G - A, N(A))$;
**else return** $\{A\} \cup ms(G - \overline{N}(A))$
**if** $A$ dominates $B$ **then return** $ms(G - B)$;
(7) $\max_1 := |ms(G - B)|$; $\max_2 := 1 + |ms(G - \overline{N}(B))|$;
**if** $\max(\max_1, \max_2) = \max_1$ **then return** $ms(G - B)$;
**else return** $\{B\} \cup ms(G - \overline{N}(B))$;
**end** {of $ms$}

The auxiliary function $ms^1(G, S)$:
**function** $ms^1(G : graph;\ S : vertexset)$: vertexset;
$\{S = \{s_1, s_2 \,|\, d(s_1) \leq d(s_2)\}\}$
**begin**
**if** $d(s_1) \leq 1$ **then return** $ms(G)$;
**if** $edge(s_1, s_2)$ **then**
**if** $d(s_1) \leq 3$ **then return** $ms(G)$
(8) **else** $\max_1 := |ms(G - \overline{N}(s_1))|$;
$\max_2 := |ms(G - \overline{N}(s_2))|$;
**if** $\max(\max_1, \max_2) = \max_1$ **then**
**return** $\{s_1\} \cup ms(G - \overline{N}(s_1))$;
**else return** $\{s_2\} \cup ms(G - \overline{N}(s_2))$;
**if** $N(s_1) \cap N(s_2) \neq \emptyset$ **then**
**return** $ms^1(G - N(s_1) \cap N(s_2), S)$;
**if** $d(s_2) = 2$ **then**
**begin**
$E, F$ :=the elements of $N(s_1)$;
{independent sets to be considered contains $s_1$ or ($s_2$, $E$ and $F$)}
(9) **if** $edge(E, F)$ **then return** $\{s_1\} \cup ms(G - \overline{N}(s_1))$;
(10) **if** $N(E) + N(F) - s_1 \subset N(s_2)$
**then return** $\{E, F, s_2\} \cup ms(G - \overline{N}(s_1) - \overline{N}(s_2))$;
{$\overline{N}(s_1) + \overline{N}(s_2)$ has no 4 element independent set containing $s_1$ or $s_2$
and $[E, F, s_2]$ dominates every other 3 element independent set}
(11) $\max_1 := 1 + |ms(G - \overline{N}(s_1))|$;
$\max_2 := 3 + |ms(G - \overline{N}(E) - \overline{N}(F) - \overline{N}(s_2))|$;
**if** $\max(\max_1, \max_2) = \max_1$ **then**
**return** $\{s_1\} \cup ms(G - \overline{N}(s_1))$;
**else return**
$\{E, F, s_2\} \cup ms(G - \overline{N}(E) - \overline{N}(F) - \overline{N}(s_2))$;
**end;**
(12) $\max_1 := |ms(G - \overline{N}(s_2))|$;
$\max_2 := |ms^2(G - \overline{N}(s_1) - s_2, N(s_2))|$;
**if** $\max(\max_1, \max_2) = \max_1$ **then**
**return** $\{s_2\} \cup ms(G - \overline{N}(s_2))$;
**else return** $\{s_1\} \cup ms^2(G - \overline{N}(s_1) - s_2, N(s_2))$;
**end** {of $ms^1$}

The auxiliary function $ms^2(G, S)$
**function** $ms^2(G : graph;\ S : vertexset)$: vertexset;
**begin** {$ms^2$. The elements of $S$ are $s_1, s_2, \ldots$ with $d(s_i) \leq d(s_{i+1})$}
(13) **if** $|S| \leq 1$ **then return** $\{\emptyset\}$;
(14) **if** $|S| = 2$ **then if** $edge(s_1, s_2)$ **then return** $\{\emptyset\}$;
(15) **else return** $\{s_1, s_2\} \cup ms(G - \overline{N}(s_1) - \overline{N}(s_2))$;
**if** $|S| = 3$ **then**
**begin**
(16) **if** $d(s_1) = 0$ **then**
**return** $\{s_1\} \cup ms^1(G - s_1, S - s_1)$;
(17) **if** $edge(s_1, s_2)$ and $edge(s_2, s_3)$ and $edge(s_3, s_1)$ **then return** $\{\emptyset\}$;
**if** $edge(s_i, s_j)$ and $edge(s_i, s_k)$ $(j \neq k)$ **then**
(18) **return** $\{s_j, s_k\} \cup ms(G - \overline{N}(s_j) - \overline{N}(s_k))$
(19) **if** $edge(s_i, s_j)$ **then**
**return** $\{s_k\} \cup ms^1(G - \overline{N}(s_k), [s_i, s_j])$;
{independent set cannot contain $s_i$ and $s_j$ and so contains one of them and $s_k$.}
**if** vertex $v \in N(s_i) \cap N(s_j) (i \neq j)$ **then return** $ms^2(G - v, S)$;
{independent set contains $s_i$ and $s_j$ and so not $v$.}
(20) **if** $d(s_1) = 1$ **then**
**return** $\{s_1\} \cup ms^1(G - \overline{N}(s_1), S - s_1)$;
(21) $\max_1 := 1 + |ms^1(G - \overline{N}(s_1), S - s_1)|$;
$\max_2 := 2 + |ms^2(G - \overline{N}(s_2) - \overline{N}(s_3) - s_1, N(s_1))|$;
**if** $\max(\max_1, \max_2) = \max_1$ **then**
**return** $\{s_1\} \cup ms^1(G - \overline{N}(s_1), S - s_1)$;
**else return**
$\{s_2, s_3\} \cup ms^2(G - \overline{N}(s_2) - \overline{N}(s_3) - s_1, N(s_1))$;
**end** $\{|S| = 3\}$;
**if** $|S| = 4$ **then**
**if** $G$ has a vertex of degree $\leq 3$ **then return** $ms(G)$
(22) $\max_1 := 1 + |ms(G - \overline{N}(s_1))|$;
$\max_2 := |ms^2(G - s_1, S - s_1)|$;
**if** $\max(\max_1, \max_2) = \max_1$ **then return** $\{s_1\} \cup ms(G - \overline{N}(s_1))$;
**else return** $ms^2(G - s_1, S - s_1)$;
**return** $ms(G)$
**end** {of $ms^2$}

## IV. AN EXAMPLE OF SOLVING THE PROBLEM OF THE MAXIMUM INDEPENDENT SET

Let $G = (V, E)$ is an undirected graph defined by the adjacency matrix $M$ :

$$M = \begin{Vmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{Vmatrix}.$$

Among all the vertices of a graph $G$ choose a vertex with the smallest degree:

$$A := 1, d(A) = 3 \text{ is minimal.}$$

In this case it is required to calculate the values of variables $\max_1$ and $\max_2$ :

$$\max_1 := |ms^2(G-1, N(1))|, \max_2 := 1 + |ms(G - \overline{N}(1))|.$$

If $\max_1 > \max_2$ then

$$ms(G) := ms^2(G-1, N(1))$$

else

$$ms(G) := \{1\} \cup ms(G - \overline{N}(1)).$$

Denote

$$G_1^1 := G - 1, S_1^1 := N(1)$$

and define the value of variable

$$\max_1 = |ms^2(G_1^1, S_1^1)|.$$

Since

$$|S_1^1| = 3 \ \ (S_1^1 = \{2, 3, 4\})$$

and

$$(3, 2) \in E, (3, 4) \in E, (2, 4) \notin E$$

then function $ms^2(G_1^1, S_1^1)$ returns the set

$$\{2, 4\} \cup ms(G_1^1 - \overline{N}(2) - \overline{N}(4)).$$

Let

$$G_2^1 := G_1^1 - \overline{N}(2) - \overline{N}(4).$$

Define the value function $ms(G_2^1)$. Since $G_2^1$ is empty graph then $ms(G_2^1) := \emptyset$, therefore,

$$ms^2(G_1^1, S_1^1) := \{2, 4\} \cup \emptyset = \{2, 4\}, \ \max_1 := |\{2, 4\}| = 2.$$

Denote

$$G_1^2 := G - \overline{N}(1)$$

and define the value of variable

$$\max_2 = 1 + |ms(G_1^2)|.$$

Among all the vertices of a graph $G_1^2$ choose a vertex with the smallest degree:

$$A_1^2 := 5, d(A_1^2) = 1 \ \text{ is minimal.}$$

In this case it's required to define the values of $ms(G_1^2)$ :

$$ms(G_1^2) := \{5\} \cup ms(G_1^2 - \overline{N}(5)).$$

Let

$$G_2^2 := G_1^2 - \overline{N}(5).$$

The function $ms(G_2^2)$ returns the set $\{7\}$ since graph $G_2^2$ consists of one vertex. Therefore, we have

$$ms(G_1^2) := \{5\} \cup \{7\} = \{5, 7\}, \max_2 := 1 + |\{5, 7\}| = 3,$$

$$\max_2 > \max_1, ms(G) := \{1\} \cup \{5, 7\} = 1, 5, 7.$$

Thus maximum independent set is $\{1, 5, 7\}$.

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